USING A VIRTUAL MANIPULATIVE ENVIRONMENT TO SUPPORT STUDENTS' ORGANIZATIONAL STRUCTURING OF VOLUME UNITS

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In this study, we investigated how Grade 3 and 4 students' organizational structure for volume units develops through repeated experiences with a virtual manipulative for building prisms. Our data consist of taped clinical interviews within a micro-genetic experiment. We report on student strategy development using a virtual manipulative for counting cubes as a measure of prism volume. A descriptive case of one student, Jim, is included as an example of how students developed increasingly efficient counting strategies built on understanding of structuring, composite units, multiplicative thinking and an understanding of the number of cubes along an edge. We found students were able to develop structure for volume and advance their level of thinking along a learning trajectory for volume measure.

Keywords: Geometry and Geometrical and Spatial Thinking, Learning Trajectories, Measurement, Technology.

Introduction and Theoretical Framework

Volume measurement is an important component of elementary school mathematics; however, geometric and spatial structuring is a topic that presents students with significant challenge (Barrett, Clements, & Sarama, 2017). Unlike length or area measurement, students must coordinate the measurements from three dimensions to measure volume. Spatial thinking is related to enumeration strategies as children measure volume (Battista & Clements, 1996). Thus, it is important to characterize the development of enumeration strategies. We have established hypothetical learning trajectories to characterize such development and to support the development of curriculum and enhance teacher knowledge (Barrett et al., 2017). A trajectory includes the mathematical learning goal, the thinking and learning in which students might engage and the pertinent learning activities to support growth from one level to the next (Simon, 1995). We employed a trajectory to classify student growth patterns and to design and test a learning activity. In this study, we designed and tested a virtual manipulative intended to support organizational structuring of volume units.

Students initially attempt to organize and structure volume units by working without coordinating the set of cube faces on the prism. We now describe this level of thinking as *volume unit repeater relater* (VURR), a fourth of seven hypothetical levels in that trajectory (Barrett et al., 2017). As children gain capability for coordinating spatial components, they see the array as space filling (i.e., a child may rightly predict an entire collection of rows to fill a layer given only one visible row). This level of thinking is described as *volume initial composite 3D structurer* (VICS), the fifth level. Once children completely integrate a set of locally coordinated structures within a global structure, they use layering strategies to enumerate the volume of a prism. This level is called *volume 3D row and* *column structure*r (VRCS), the sixth level of the trajectory. There is still a need to expand our knowledge of effective instructional interventions for any given level in this trajectory. We developed an intervention intended to bring students up to the VRCS level. Battista and Clements (1998) found that most grade five students, but only 20% of grade three students, characterized a rectangular prism as a series of layers of rows and columns of cubes. Thus, we focused on grades three and four.

We developed and used a computer manipulative, our intervention, to help students recognize that edge lengths can be used to predict the number of cubes along an edge, and develop their use of composite units or units of units. We hypothesized that the manipulative would highlight the efficiency of enumerating composite units by constraining how students interact with increasingly complex units: first single cubes, then row collections of cubes, then layered collections of rows of cubes. Further, we expected to prompt for a correspondence between an edge length and the number of an appropriate unit along that edge. The following research questions guided our investigation:

- How does a student's organizational structure of volume units change through repeated experiences with a virtual manipulative for building prisms?
- What are the critical features of the treatment that supported student development?

Methodology

In this study we investigated the thinking of 31 participants in Grade 3 (14 students) and Grade 4 (17 students) at a private school in the Midwest. We used a microgenetic method to focus on growth. Three aspects of this provide insight on growth: (a) observations that span the whole period of rapidly changing competence, (b) the density of observation within this period is high, relative to the rate of change; and (c) observations of changing performance are analyzed intensively to indicate the processes that give rise to them (Siegler & Sventina, 2006, p. 1000). All 31 students participated in five interviews with one or two researchers. During each interview (taking approximately 15 minutes), the students completed three trials. A trial included first a paper version of a prism to measure, and then a virtual manipulative of the same prism.

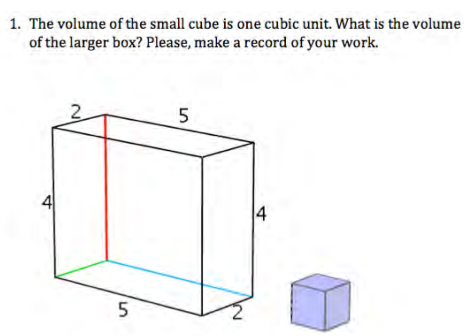


Figure 1. Paper portion of trial**.**

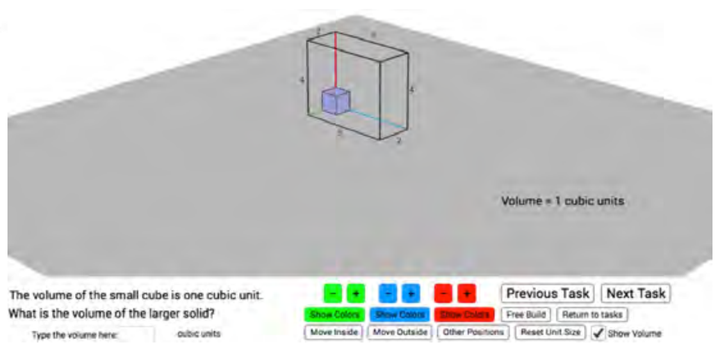
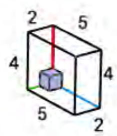
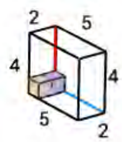
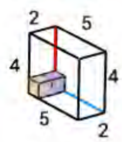
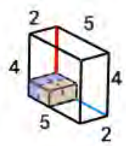
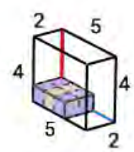
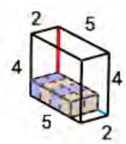
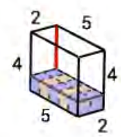
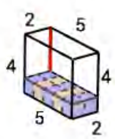
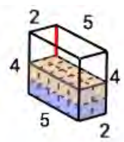
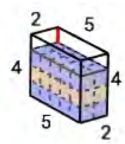
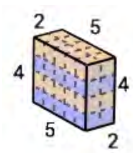


Figure 2. Virtual manipulative of trial**.**

For each trial, students had to find the volume of a rectangular prism (see Figure 1) using a pencil and a paper showing a prism with some edges labeled for length (in cm). The interviewer asked, "The volume of the small cube is one cubic unit. What is the volume of the larger solid?" After the student completed the paper task, they were asked to predict the volume of the same rectangular prism using a virtual manipulative. Each student was prompted to predict the next outcome of a button sequence, first for width, then length and finally altitude. They were asked, "How many cubes will you have when you are done pressing the green button [the first button sequence, width of prism]?" The green button sequence produced one complete row. Likewise, we asked students to predict for the second button sequence (the blue button; forming one layer of rows, now width and length), and for the third button sequence (red button; the total volume) (see Figure 2) if they continued making correct predictions about the accumulation of cubes and groups of cubes. If not, the interviewer directed them to press the relevant button so they would accumulate the cubes along that current dimension; this produced a report of the number of cubes associated with the button sequence they had been predicting. Our approach merely hinted that the prediction had been incorrect. Working thus, in three stages, students gradually filled the rectangular prism shown on a computer screen (see Table 1). (The virtual manipulative can be found at: https://www.geogebra.org/m/FgQVdDTb).

Table 1: Virtual Manipulative Program

	Click 0	Click 1	Click 2	Click 3	Click 4
First button sequence (green)			NA	NA	NA
Second button sequence (blue)					
Third button sequence (red)					NA

Results and Discussion

Our research question focuses on changes in students' organizational structure of volume units, so we categorized our participants into three categories to select participants who had changed their structuring. Of the 31 participants, eight (two Grade 3 and six Grade 4) showed prior, adequate knowledge of the structuring of volume and thus were not positioned to benefit during the study. Of the remaining 23 participants, eight (five Grade 3 and three Grade 4) did not display any meaningful changes in their ability to leverage the structure of 3D arrays to find volume. This left us with 15 (seven Grade 3 and eight Grade 4) participants out of a possible 23 (65%) who demonstrated changes in their organizational structuring of volume. The fact that 65% of the participants who could have benefited from this study suggests the virtual manipulative helped them measure prism volume by emphasizing and portraying structured sequences of units. To find the nature of the changes and relate them to their experiences, we describe a case study of one student. We anticipated finding a correspondence between changes in structuring the groups of units and the salient features of the virtual manipulative.

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Case Study: Jim

Jim, a fourth-grade student, demonstrated the VICS and VRCS levels across the eleven trials. We trace his development of increasingly sophisticated use of units and understanding of length labels. We begin the case study by sketching our own model of his strategic interaction with the interviewer and the tasks and the given tools: initially, Jim's structuring included rows as units but the number of units in a row was not connected to the labels. Trial 4 was the first time he incorporated the length labels to guide his structuring of units. Next, he extended his structuring to include rows as units guided by the length labels, and then he extended his structuring to include layers as units. Finally, Jim's skip counting evolved into multiplication as a scheme for finding volume of rectangular prisms. Next, we offer a detailed interpretation across four trials. Lastly, we interweave observations and interpretations to present an ongoing model of Jim's thinking in keeping with our theoretical perspective drawn from the learning trajectory of volume measurement.

On the paper portion of Trial 1 (2 by 5 by 4 rectangular prism; see Figure 1), Jim referred to the unit cube as one, pointing to it. Next, he mentioned five, "it was five long, right?" dragging his finger across the bottom front edge (edge labeled 5). Jim then dragged his finger up the image of the prism skip counting, "five, ten, fifteen, twenty, twenty-five." He then said, "twenty-five of those" pointing back to the unit cube. We note that on the first trial Jim was already skip counting, an indication that he was treating a collection of five as a repeatable unit. However, he counted five sets of five, which was inconsistent with the length labels of 5 and 4. Additionally, he appears to have only dealt with two of the dimensions to arrive at his final answer of 25.

On the computer portion of Trial 1 Jim moved the unit cube inside the prism. He pointed with the cursor to the cube and said, "one". He then pointed to the next place he expected a cube would fit and said "two". He paused and continued in a regular pattern: "3, 4,…, 5, 6, …, 7, 8, …, 9, 10". Each pause included a motion to the next position on the base of the prism. We take this pattern in his counting and motion as an indication that although he was still counting single units. He was also attending to groups of two units to fill the bottom layer of the prism.

Still, prior to pushing any buttons (i.e., he could see just one cube inside the prism corner), he moved the cursor along the bottom front edge and the bottom back edge (edge labeled 5) saying, "so five, five". Then he took the cursor as he had moved his finger earlier on the paper portion and moved up successively on the front face. This time he skip counted by tens instead of fives and said, "So it would be 50 not 25. Ah, that is 25 times 2." His actions and statements indicate to us his attention to five countable entities during this trial: a unit cube, a row of two cubes, and a row of 5 cubes, horizontal layers of 10 cubes and vertical layers of 25 cubes. Although he exhibited an understanding of composite units, he did not mention or make use of the three distinct length labels. He treated the height as if it were five units high, even though that edge was labeled "4". We believe he was counting by visual estimation and repeated pointing gestures to find the height. This seems to indicate the VICS level because he attended to unit cubes as parts of rows, and rows as parts of layers, yet he was not using all three dimension labels to structure his groups (as he would if he operated at the VRCS level).

Continuing with the computer portion of Trial 1, the researcher asked Jim to predict how many cubes he would have when he finished with the first button sequence (colored green). He answered correctly, "two". Next the researcher asked him to predict the number of cubes after pressing the second button sequence (blue). Note that the second button was designed to add additional rows along the second dimension of the prism. Jim's response did not match that of the computer environment, "There will be five (motioning along the blue line) plus one is six. There is going to be six blocks." Because he used the cursor to touch five points along the back edge, and then one in the front left corner, he counted six cubes that included a row plus one cube next to that row. Next, Jim clicked through the first button sequence (green), and stopped with two cubes (to form a row). Thus,

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his prediction of two matched the number of cubes displayed. At this moment, Jim said, "wait no it adds two times five. The blue is going to add four more." We believe he meant four more groups of two cubes each. The interviewer asks him how many he would have when he finished using the blue button (which is the second button sequence) to add cubes. He said, "ten" (the correct number of cubes following the blue button sequence). We infer that he was creating a more sophisticated approach to counting composite collections of units that incorporated multiplicative operations. He was counting not only single cubes, but by coordinating the two dimension labels, of 2 and 5 along the base edges, he was able to use multiplication to group 10 cubes as five sets of two.

The researcher then asked him how many cubes he thought he would have when he finished with the red button (the third button sequence that builds vertically). Jim said he thought that it would take 30 for the whole box to be full. If he was thinking he would have to add 30 more, he was correct, but 30 cubes was not the total. Now the researcher asked him to use the blue button (the second sequence). While clicking along the blue line, Jim stated that he was wrong because it only added two and he thought he would click it once to add eight. We believe Jim was making sense of the program and the structuring of volume. Once he finished pressing the blue buttons (the second sequence), Jim was asked to predict how many cubes there would be when he was finished with the red button (the third button sequence). Jim said there would be fifty, which is incorrect. Nevertheless, after one click of the red button with 20 cubes showing, Jim said, "no, it only adds the ten more." Jim then went on to say, "now I get it, each one adds one more, like times two." We think this statement influenced his later work and solution on the paper portion of Trial 3. Once Jim filled the box, he said that the answer was forty but said he did not know for sure. Recall his work on the paper portion and his first two predictions on this computer trial were 25, and 50 cubes, rather than 40. To sum up, we think Jim relied on visual estimates to find the number of cubes in the rows during trial 1. Alternatively, he may have reported the number of square faces on the front of the prism.

Moving on to trial 2, we expected improved performance from Jim, particularly on the computer portion, as he had now practiced using the sequence of three buttons. Also, we anticipated that Jim would build on his use of composite units evidenced by his skip counting and multiplication strategy for counting row groups in Trial 1. On the paper portion of Trial 2 (5 by 4 by 3 rectangular prism), Jim drew a cube inside the rectangular prism, mimicking the virtual manipulative. He then said that it was four across, three up, consistent with the length labels. He counted four across and then said twenty; we take this as evidence that he was treating a row of five as a composite unit, and four of these rows would be consistent with the bottom layer of the rectangular prism. Next, he said "twenty times three, twenty times two, forty. I think it is forty." We notice a shift in his use of units, as he is now counting groups of twenty. The researcher then told Jim that if he needed help with calculations he could help. Jim responded, "Then you go one, two, three, four, five, that is four times five that is twenty. Then you already have one floor done, then you times two, that is forty. That fills the full box." Here we take his counting, one, two, three, four, five as another instance of his counting composites, five sets of fours, which gives him twenty; we also interpret his actions as a way of unitizing the twenty as "one floor".

On the computer portion of Trial 2, Jim correctly predicted the first button sequence and the second button sequence (he finds 20 on the floor layer), but not the third (for the whole prism). He said, "There will be thirty [in the whole prism]." Next, Jim used the virtual manipulative to fill the rectangular prism (the computer screen showed: *volume = 60 cubic units*). When he finished, he said, "No, 60…. I know why I am wrong." The interviewer asked Jim what he thought the actual volume was. After a pause, Jim said, "I don't know. I am thinking 60 cubic units and I did 40 cubic units on the last one (he refers to his answer on paper). I am thinking 60 but I don't know. It might be, hum… I think it is 60." We believe Jim is still operating at the VICS level because he has now begun coordinating spatial components more widely, but has not yet completely integrated the set of local structures into a global structure. His ability to identify a layer without finishing the structure to find 60 cubes indicates a transitional state of thinking.

Next, on the paper portion of Trial 3 (3 by 3 by 4 rectangular prism), Jim said, "4, 3, 3… now I get it." While he said that he pointed to the length labels of each number. Next, he said, "that would be nine on the bottom… nine, nine plus nine is 18… 18 plus 9 is …" The researcher told him that was 27. Then Jim said, "I have to add 27 more. Twenty-seven plus 27." He said, "44, …, no, 54." Then he said, "I am going with 54. I hope I get this right." The correct answer was 36. During, this portion of Trial 3, we believe Jim showed a development in composite units; he found the bottom layer of nine and then used that number of cubes to accumulate another unit of nine, perhaps another layer. He appeared to skip count, adding 9's, but when he reached the third layer he doubled that quantity of 27, reaching 54. Why did he do this? We believe he followed a pattern he had set in his work on the virtual manipulative for Trial 1 when he had explained moving from one layer to two as a doubling. Because it had been the first layer, a doubling process was equivalent to adding one layer. But he did not try this doubling approach with higher levels until this instance, and here he doubles the third layer sum of 27 to get 54. He appears to intend this 54 as the accumulation of the fourth and final layer.

For the computer portion of Trial 3, Jim correctly predicted the first button sequence and the second button sequence, giving a correct prediction for the number of cubes in the bottom layer, similar to his paper task work. Next, Jim was asked how many cubes there would be when he was finished with the red button (third button sequence). He said, "it goes up 9, 18, wait… 27, 27 plus 9… I know it is 54." As Jim filled the rectangular prism with the red button sequence, he skip counted by nine. After the prism was completely filled with cubes, Jim explained, "I went too many high because for 54. I added 18 more at 27 because 27 plus 9 is 36 … I know why I keep screwing it up." Next, he said that he thought the box (pointing to the unit cube) was smaller and he "did tiny ones".

We believe this experience with trial 3 was transitional for Jim and this is where he made a connection between the length label for the height of the rectangular prism and the number of layers. When Jim was asked what the actual volume was, he said it was 36 cubic units without hesitation as before. We believe Jim was transitioning toward VRCS in Trial 3, but still operating at the VICS level. Next we describe trial 4 which took place at the beginning of the second interview.

On the paper portion of Trial 4 (3 by 5 by 4 rectangular prism), Jim paused his work on paper to comment how he thought the virtual manipulative (computer task) worked. He said (pointing to the left bottom of his paper prism), "There are three here" and he made three marks with his pen (see Figure 3). While pointing to the computer, he said, "You hit the green, you would get one more, no wait, you would get two more." Next, he wrote down 3 units along the green line. Following he said, "The blue, you would get three more" and he wrote that down on the page along the blue line. Next, he said, "The red goes up one more but adds three" and wrote that on his page along the red line. Then he said, "So you get three (pointing along the green line) and then three more and three more, no that is six, six more and then you will have twelve." The researcher asked him if he thought 12 was the total volume and Jim responded, "No, no, you have twelve altogether right there in that little square" pointing to the 3x4 vertical face. The researcher asked, "What happens after that?" Jim responded, "You go, 3 and 5 (pointing to the base of the prism) that is 15, 15 + 15 is 30, wait no… 15 so, 15 (drawing one mark under the prism), 30,… two (drawing another mark below his prior), um 45 is three (drawing another mark), and then 50, no, 60 is four, yeah. So there is going to [be] 60 blocks in all." On this trial, Jim integrated a set of locally coordinated structures within a global structure, and used a layering strategy to enumerate the volume of a prism. We interpret this incident to mean Jim is operating at the VRCS level for measuring volume. He coordinates sets of cubes as rows and groups of rows as layers.

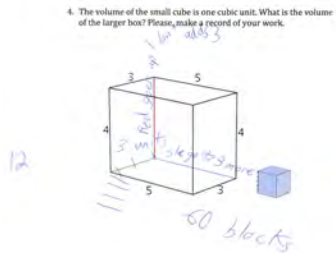

Figure 3. Jim’s paper portion of Trial 4.

On the computer portion of Trial 4, Jim correctly predicted the number of cubes for each button sequence. After he filled the prism, Jim appeared very excited, “Got that one right!” Jim is demonstrating thinking consistent with the VRCS level. On the rest of the Trials (5-11), Jim completed the paper and computer predictions correctly. As he worked on the paper portions, he talked about how the computer program would work. At the end of Trial 5, Jim said he could use multiplication. He explained that you just do the six times the five, times the two. Jim showed another advancement of thinking. He shifted from skip counting to multiplying to predict the number of cubes. However, on the Trial 6 Jim used repeated addition again to find the answer. He said he was surprised he had the right answer because, “threes and sixes are hard for multiplication.” Beyond Trial 6, Jim mostly used multiplication to multiply three edge lengths.

Conclusions and Implications

Measuring volume is a challenge for students (Barrett et al., 2017). We found that only six of the 31 participants in the study exhibited prior knowledge of spatial structuring for measuring the volume of rectangular prisms. Fifteen of the 23 participants who could have benefited from the experiences in this study developed more effective strategies and answered with increasingly correct measures. We take this as evidence that the treatment in this study was effective in guiding students to build a more structured understanding of volume.

We have used Jim’s case to represent many students who did not initially use the length labels to predict the number of units fitting an edge. It was not until Trial 4 that Jim first used length labels to predict the number of units along an edge. Specifically, students developed more efficient enumeration strategies as well as a meaningful interpretation of the length labels as a way to predict the number of volume units fitting along an edge. The effectiveness of this treatment suggests an intervention to complement the volume learning trajectory we used to design the treatment (Barrett et al., 2017). The features of the treatment that supported student growth were those that prompted students to associate length measure labels on the prism edges with 3D arrays and those that promoted the flexible use of unit groups. Thus, the repeated pairing of length measurements (length labels) with the corresponding number of volume units was an important feature in helping students discover the predictive power of the length labels. We emphasized the pairings of labels to edge length by the continual, predict-and-check questions about accumulating quantity, through various unit groups (i.e., “How many will there be when you are done with the blue button?”). Second, students operated the three sequences of buttons independently, to meet goals of filling along three different but related dimensions of the prism. Their actions resulted in predictable but different numbers of additional cubes filling out various prism cases. We believe this was an important feature in guiding students to develop more efficient enumeration strategies. Students were expected to cope with a single button press resulting in three possibilities: the addition of a single cube, a row of cubes, or a layer of cubes, depending on the sequence. The pairing of a single action (a button press) with the appearance of either a 1D, 2D or 3D array of units suggested to students the value and utility of grouping units, and of the global coordination of those units.

In summary, we claim that changes in students' organizational structure of volume units followed from their experience with particular aspects of the virtual manipulative we employed. Secondly, we found students learned organizational structure of volume units through developing a flexibility of single units and composite units (i.e., single unit cubes, rows as units, and layers as units). Lastly, students enhanced their strategies for calculating volume by transitioning from repeated addition to multiplicative thinking.

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References

Barrett, J. E., Clements, D. H., & Sarama, J. (2017). Children's measurement: A longitudinal study of children's knowledge and learning of length, area, and volume. *Journal for Research in Mathematics Education Monograph*. Vol. 16. Reston, VA.

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education, 27,* 258−292.

Battista, M. T., & Clements, D. H. (1998). Students' understanding of three-dimensional cube arrays: Findings from a research and curriculum development project. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 227–248). Mahwah, NJ: Lawrence Erlbaum Associates.

International GeoGebra Institute. (2015). GeoGebra (Version 5.0.42) [Software]. Available from www.geogebra.org

Siegler, R. S., & Svetina, M. (2006). What leads children to adopt new strategies?: A microgenetic/cross-sectional study of class inclusion. *Child Development*, *77*(4), 997–1015. doi:10.1111/j.1467-8624.2006.00915.x

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145. doi:10.2307/749205